Function theory on Kaehler manifolds

Note by Man-Chun LEE

1 Kaehler manifolds

Let M^n be a smooth manifold. A riemannian metric g is a smooth section of $T^*M \otimes T^*M$ such that g is symmetric and positive definite at any $p \in M$. In local coordinate $(x_1, ..., x_n)$,

$$g = g_{ij}\, dx^i \otimes dx^j.$$

In addition, if M is a complex manifold with almost complex structure $J \in TM \otimes T^*M$ and $g(X, Y) = g(JX, JY)$, $\forall X, Y \in TM$. Then M is called Hermitian manifold, (M, g, J).

One can define a 2-form ω_g where $\omega_g(X, Y) = -g(X, JY)$. Let ∇ be the Levi-connection of a Hermitian manifold (M, g, J).

Definition 1.1. *A Kahler manifold (M, g, J) is a Hermitian manifold such that $\nabla J = 0$. In particular, (M, J) is a complex manifold.*

Proposition 1.1. $\nabla J = 0$ *if and only if* $d\omega = 0$. *Thus, a Kahler manifold is also a symplectic manifold.*

Proof. For $X, Y, Z \in \Gamma(TM)$. By invariant formula, we have

$$\begin{aligned} d\omega(X, Y, Z) &= -Xg(Y, JZ) + Yg(X, JZ) - Zg(X, JY) \\ &\quad + g([X, Y], JZ) - g([X, Z], JY) + g([Y, Z], JX) \end{aligned}$$

We choose $X = e_1, Y = e_2, Z = e_3$ to be normal coordinate vector field at p in order to simplify our calculation. By $J^2 = -Id$, we have at p,

$$\begin{aligned} d\omega(X, Y, Z) &= -g(Y, (\nabla_X J)Z) - g(X, (\nabla_Z J)Y) + g(X, (\nabla_Y J)Z) \\ &= g(Z, (\nabla_X J)Y) + g(Y, (\nabla_Z J)X) + g(X, (\nabla_Y J)Z). \end{aligned}$$

The above equation is independent of choice of coordinate, so it is valid for arbitrary X, Y, Z. At the same time, since $J\nabla J + \nabla J \cdot J = 0$,

$$\begin{aligned} g((\nabla_X J)Y, Z) &= g(\nabla_X(JY), Z) - g(J\nabla_X Y, Z) \\ &= Xg(JY, Z) - g(JY, \nabla_X Z) - g(J\nabla_X Y, Z) \\ &= -X\omega(Z, Y) + \omega(\nabla_X Z, Y) + \omega(Z, \nabla_X Y) \end{aligned}$$

Because the connection is torsion free, using invariant formula

$$2g((\nabla_X J)Y, Z) = d\omega(X, Y, Z) - d\omega(X, JY, JZ).$$

$\square$

1.1 Curvature on Kahler manifold

Given a Kahler manifold (M, g, J), extend g $\mathbb{C}$-linearly to $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$.

Proposition 1.2. *If u, v are both in $T^{1,0}M$ or $T^{0,1}M$, then $g(u, v) = 0$.*

Thus, locally we have

$$\omega_g = \frac{\sqrt{-1}}{2} \sum_{i,j} g_{i\bar{j}}\ dz^i \wedge d\bar{z}_j.$$

Denote $h(u, v) = g(u, \bar{v})$ for $u, v \in T^{1,0}M$, then h becomes a hermitian inner product.

Extend ∇ in a $\mathbb{C}$ way to $\Gamma(T_{\mathbb{C}}M)$.

$$\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j} = \Gamma_{ij}^{\bar{k}} \frac{\partial}{\partial z^{\bar{k}}} + \Gamma_{ij}^{k} \frac{\partial}{\partial z^k}, \quad \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^{\bar{j}}} = \Gamma_{i\bar{j}}^{\bar{k}} \frac{\partial}{\partial z^{\bar{k}}} + \Gamma_{i\bar{j}}^{k} \frac{\partial}{\partial z^k}.$$

Because of the fact that $\nabla J = 0$, this will imply

$$\Gamma_{ij}^{\bar{k}} = \Gamma_{i\bar{j}}^{\bar{k}} = \Gamma_{i\bar{j}}^{k} = 0.$$

So the only non-vanishing term will be $\Gamma_{\bar{i}\bar{j}}^{\bar{k}}$ and Γ_{ij}^{k}.

By definition,

$$\frac{\partial g_{j\bar{k}}}{\partial z^i} = g(\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^{\bar{k}}}) + g(\frac{\partial}{\partial z^j}, \nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^{\bar{k}}}) = \Gamma_{ij}^{l}\ g_{l\bar{k}}.$$

So we have the following formula.

$$\Gamma_{ij}^{l} = g^{l\bar{k}} \partial_i g_{j\bar{k}}.$$

Proposition 1.3. *M is a Kahler manifold if and only if for all $x \in M$, we can find a holomorphic chart $(z_1, ..., z_n)$ such that $g_{i\bar{j}}(x) = \delta_{ij}$ and $dg_{i\bar{j}}(x) = 0$.*

Recall Riemannian curvature tensor:

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w \quad , u, v \in T_{\mathbb{R}}M.$$

Extend Rm complex linearly to $T_{\mathbb{C}}M$ and denote $R(u, v, w, x) = g(R(u, v)w, x)$.
Since J is parallel, $R(u, v)(Jw) = J(R(u, v)w)$. Thus,

$$R(u, v, Jw, Jx) = R(u, v, w, x).$$

By symmetric of curvature operator, $R(u, v, w, x) = 0$ if w, x are of same type. The only non-vanishing term are $R_{i\bar{j}k\bar{l}}$.

By first Bianchi identity, we have a extra symmetric property for Rm on Kahler manifolds.

$$R_{i\bar{j}k\bar{l}} + R_{i\bar{l}\bar{j}k} = 0.$$

Proposition 1.4. *In local coordinate,*

$$R_{i\bar{j}k\bar{l}} = -\partial_k\partial_{\bar{l}}g_{i\bar{j}} + g^{s\bar{t}}\partial_k g_{s\bar{j}} \cdot \partial_{\bar{l}}g_{i\bar{t}}.$$

We define Ricci curvature on M by

$$R_{k\bar{l}} = g^{i\bar{j}}R_{i\bar{j}k\bar{l}} = -\partial_k\partial_{\bar{l}} \log det(g_{i\bar{j}}).$$

Some notation for complex manifold:
For $f \in C^\infty(M)$, $d = \partial + \bar{\partial}$ where

$$\partial f = \sum_i \frac{\partial f}{\partial z^i}dz^i, \;\; \bar{\partial} f = \sum_i \frac{\partial f}{\partial \bar{z}^i}d\bar{z}^i.$$

In general,

$$\partial : A^{p,q} \to A^{p+1,q}, \;\; \bar{\partial} : A^{p,q} \to A^{p,q+1}.$$

And $\partial, \bar{\partial}$ satisfies $\partial^2 = \bar{\partial}^2 = 0, \; \partial\bar{\partial} + \bar{\partial}\partial = 0$.

The Ricci form is then given by

$$Ric = \frac{\sqrt{-1}}{2}R_{i\bar{j}}dz^i \wedge dz^{\bar{j}}.$$

So locally

$$Ric = -\frac{\sqrt{-1}}{2}\partial\bar{\partial} \log \det g$$

which is clearly a closed, real (1,1) form.
Remark: One can check that the definition is same as the Ricci curvature in Riemannian case. In particular, if we choose a orthonormal base $\{e_1, ..., e_n\}$ such that $Je_i = e_{n+i}$. And $u_i = (e_i - \sqrt{-1}e_{n+i})/\sqrt{2}$. Then $\{u_i\}$ is a unitary frame on $T^{1,0}M$. Direct computation yield

$$Ric(u_i, \bar{u}_i) = Ric(e_i, e_i).$$

Definition 1.2. *We say M has a non-negative bisectional curvature, $BK \geq 0$ if*

$$R(u, \bar{u}, v, \bar{v}) \geq 0 \;\; \textit{for all}\, u, v \in T^{1,0}M.$$

We say M has a non-negative holomorphic curvature, $H \geq 0$, if

$$R(u, \bar{u}, u, \bar{u}) \geq 0 \;\; \textit{for all}\, u \in T^{1,0}M.$$

Noted that $BK \geq 0$ imply $Ric \geq 0$. But the relationship between H and Ric remains unknown.

1.2 Result about structure of manifolds with curvature constraint

Theorem 1.3. *(Uniformization theorem) Every simply connected Riemann surface is conformally equivalent to one of the three domains: the open unit disk, the complex plane, or the Riemann sphere.*

Uniformization conjecture by Yau: If M is complete non-compact Kahler manifold with $BK > 0$, then M is biholomorphic to $\mathbb{C}^n$.

Theorem 1.4. *(Soul theorem) If (M, g) is a complete connected Riemannian manifold with sectional curvature $K \geq 0$, then there exists a compact totally convex, totally geodesic submanifold S such that M is diffeomorphic to the normal bundle of S.*
In particular, if $K > 0$, then S is a singleton. That is to say M being diffeomorphic to R^n.

Theorem 1.5. *(Frankel Conjecture solved by Siu-Yau, Mori) If M is compact Kahler manifold with $BK > 0$, then M is biholomorphic to $\mathbb{CP}^n$.*

Definition 1.6. *Given a Riemannian manifold M^n, let $p \in M$. We say M has maximal volume growth if $\frac{Vol(B(p,r))}{r^n} \geq c > 0$ for all $r > 0$.*

Theorem 1.7. *(Mok-Siu-Yau,1981) Let M be a non-compact Kahler manifold. If there exists $C, \epsilon > 0$ such that $0 \leq BK(x) \leq \frac{C}{r(x)^{2+\epsilon}}$ and M has maximal volume growth, then M is biholomorphic and isometric to $\mathbb{C}^n$.*

2 Mok-Siu-Yau's result

Theorem 2.1. *(Mok-Siu-Yau,1981) Let M be a non-compact Kahler manifold. If there exists $C, \epsilon > 0$ such that $0 \leq BK(x) \leq \frac{C}{r(x)^{2+\epsilon}}$ and M has maximal volume growth, then M is biholomorphic and isometric to $\mathbb{C}^n$.*

Proposition 2.1. *M^n is Kahler manifold with $BK \geq 0$. Let ρ be a real d-closed (1,1) form, $f = \rho_{i\bar{j}} g^{i\bar{j}}$. If $f = \frac{1}{2}\Delta_{\mathbb{R}} u$ for some function u, then $||\sqrt{-1}\partial\bar{\partial}u - \rho||^2$ is sub-harmonic.*

Proof. Choose a normal coordinate at p. Then $R_{i\bar{j}k\bar{l}} = -\partial_k \partial_{\bar{l}} g_{i\bar{j}}$ at p. Let $v = \sqrt{-1}\partial\bar{\partial}u - \rho$.

$$\begin{aligned}\frac{1}{2}\Delta||v||^2 &= \partial_i \partial_{\bar{i}} (g^{i\bar{l}} g^{k\bar{j}} v_{i\bar{j}} \overline{v_{l\bar{k}}}) \\ &= R_{l\bar{i}p\bar{p}} v_{i\bar{j}} \overline{v_{l\bar{j}}} + R_{j\bar{k}p\bar{p}} v_{i\bar{j}} \overline{v_{i\bar{k}}} + \left|\partial_p v_{i\bar{j}}\right|^2 + \left|\partial_{\bar{p}} v_{i\bar{j}}\right|^2 \\ &\quad + \partial_p \partial_{\bar{p}} v_{i\bar{j}} \cdot \overline{v_{i\bar{j}}} + v_{i\bar{j}} \cdot \partial_p \partial_{\bar{p}} \overline{v_{i\bar{j}}}\end{aligned}$$

By $\partial\bar{\partial}$ lemma, $\rho = \sqrt{-1}\partial\bar{\partial}w$ for some w locally as ρ is d-closed (1,1) form. By assumption, $trac(v) = 0$. Taking derivative on $trac(v)$ to yield

$$\sum_{k,l} R_{l\bar{k}i\bar{j}} v_{k\bar{l}} \overline{v_{i\bar{j}}} + \sum_k \partial_i \partial_{\bar{j}} v_{k\bar{k}} \cdot \overline{v_{i\bar{j}}} = 0, \ \ \forall\, i, j.$$

By mean of transformation of $U(n)$, we may further assume $v_{i\bar{j}} = a_i \delta_{ij}$ at p, where a_i are real. Then,

$$\begin{aligned}\frac{1}{2}\Delta||v||^2 &\geq R_{l\bar{i}p\bar{p}} v_{i\bar{j}} \overline{v_{l\bar{j}}} + R_{j\bar{k}p\bar{p}} v_{i\bar{j}} \overline{v_{i\bar{k}}} + \partial_p \partial_{\bar{p}} v_{i\bar{j}} \cdot \overline{v_{i\bar{j}}} + v_{i\bar{j}} \cdot \partial_p \partial_{\bar{p}} \overline{v_{i\bar{j}}} \\ &\geq 2R_{i\bar{i}p\bar{p}} a_i^2 - 2R_{i\bar{i}p\bar{p}} a_i a_p \geq 0.\end{aligned}$$

∎

Proposition 2.2. *$(M^n, p), n \geq 3$ is a manifold with $Ric \geq 0$ and maximal volume growth. Let $f \geq 0$ be a smooth function on M. Then $\Delta u = f$ has a solution if*

1. *$f(x) \leq \frac{C}{1+r(x)^2}$, and $\exists c_1$ such that $-c_1 \log(r(x)+2) \leq u(x) \leq c_1 \log(r(x)+2)$.*

2. *$f(x) \leq \frac{C}{1+r(x)^{2+\epsilon}}$, and $\exists C_2 > 0$ such that $|u(x)| \leq C_2$.*

Proof. Here we will only demonstrate case 1.
(Some review for Green function [5]) For M^n with $Ric \geq 0$, it is non-parabolic if and only if

$$\int_1^\infty \frac{t}{V_p(t)} < \infty.$$

Due to the non-collapsing assumption, Greeen function $G(x,y)$ exists with the following properties.

1. $G(x,y) > 0$**,**

2. $\exists\ c > 0$ **s.t.** $\frac{c^{-1}}{d^{n-2}(x,y)} \leq G(x,y) \leq \frac{c}{d^{n-2}(x,y)}$**,**

3. $G(x,y) = G(y,x)$**,**

4. $\int_M G(x,y)\Delta f(y)\, dy = -f(x)$ **for** $f \in C_0^\infty(M)$**,**

5. $|\nabla_x G(x,y)| \leq \frac{C}{d^{n-1}(x,y)}$.

For $R > 0$, let $f_R(x) = \varphi(x)f(x)$ where φ is a non-increasing smooth function in which $\varphi = 1$ on $B(p, R+1)$, $\varphi = 0$ on $B(p,R)$ and φ', φ'' are bounded. Let

$$u_R(x) = \int_M [G(p,y) - G(x,y)] f_R(y)\, dy.$$

Then u_R satisfy $\Delta u_R = f_R$ and $u_R(p) = 0$. For $x \in M$, $R >> r = r(x)$. By gradient estimate for G,

$$\begin{aligned}\int_{B(p,R)\setminus B(p,2r)} |G(p,y) - G(x,y)|\, f_R(y)\, dy &\leq \int_{B(p,R)\setminus B(p,2r)} \frac{cr}{(d(y,p)-r)^{n-1}} f(y)\, dy \\ &\leq \int_{B(p,R+1)\setminus B(p,2r)} \frac{Cr}{(1+r(y)^2)(r(y)-r)^{n-1}}\, dy.\end{aligned}$$

By volume comparsion, we further deduce that

$$\int_{B(p,R+1)\setminus B(p,2r)} |G(p,y) - G(x,y)| f_R(y)\, dy \leq \int_{2r}^R \frac{Ct^{n-1}r}{(1+t^2)(t-r)^{n-1}} dt \leq C_n.$$

On the other hand,

$$\begin{aligned}\int_{B(p,2r)} G(p,y)f(y)\, dy &\leq \int_{B(p,2r)} \frac{c}{r(y)^{n-2}(1+r(y)^2)} dy \\ &= \int_0^{2r} \frac{c}{t^{n-2}(1+t^2)} dA_t dt \\ &\leq C\log(r+2).\end{aligned}$$

$$\int_{B(x,r/2)} G(x,y)f(y)\,dy \leq \frac{C_n}{r^2+4}\int_0^{r/2} s\,ds \leq C_n.$$

Also, for $r > 0$

$$\begin{aligned}\int_{B(p,2r)\setminus B(x,r/2)} G(x,y)f(y)\,dy &\leq \int_{B(p,2r)\setminus B(x,r/2)} \frac{c}{d(x,y)^{n-2}(d(p,y)^2+1)}\,dy \\ &\leq C_1\int_{-r/2}^{2r}\frac{s+r}{s^2+1}\,ds \\ &= \frac{C_1}{2}\log\frac{16r^2+1}{r^2+4} + C_1 r\cdot[\arctan(2r)+\arctan(r/2)] \\ &\leq C_n.\end{aligned}$$

Combine all this, we have

$$|u_R(x)| \leq C_n[\log(r(x)+2)].$$

We now claim that $\lim_{R\to\infty} u_R(x)$ exists after passing to subsequence.
First as M has non-negative Ricci curvature and is non-collapsing, Sobolev inequality with compact support is valid. Furthermore, the sobolev constant is an absolute constant. Thus we have Harnack inequality for positive harmonic functions on geodesic ball and hence the Holder estimate

$$|u(x)-u(x_0)| \leq C_R|x-x_0|^\alpha$$

for $x, x_0 \in B(p,R)$.
Let $R1 < R_2 < ... < R_n < ...$ correspond the exhaustion, for each $j \in \mathbb{N}$, define $v_i = u_{R_i} - u_{R_j}$ which is harmonic on $B(p,R_j)$. Thus for $x, y \in B(p, R_j/2)$,

$$|u_{R_i}(x)-u_{R_i}(y)| \leq |v_i(x)-v_i(y)| + |u_{R_j}(x)-u_{R_j}(y)| \leq C_j|x-x_0|^\alpha + |u_{R_j}(x)-u_{R_j}(y)|.$$

So, $\{u_{R_i}\}_{i=j}^\infty$ is equicontinuous on $B(p,R_j/2)$. By ArzelAscoli theorem, we can extract convergent subsequence $\lim_{R\to\infty} u_R(x)$ on $B(p,R_j/2)$. Using diagonal argument, we have limit function $u(x) = \lim_{R\to\infty} u_R(x)$. By gradient estimate of G, we infer that $|\nabla u| = O(1/r)$. □

Lemma 2.2. *Second derivative estimate for both situations:*

$$\frac{1}{vol(B(p,R))}\int_{B(p,R)} |\nabla^2 u|^2 \leq \frac{C}{R^4} \text{ for } R > 1.$$

Proof. Recall bochner's formula,

$$\frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla\Delta u\rangle + Ric(\nabla u, \nabla u) \geq |\nabla^2 u|^2 + \langle \nabla u, \nabla f\rangle.$$

Let ϕ be a cut-off function such that $\phi = 1$ on $B(p,R)$, $\phi = 0$ outside $B(p,2r)$, and

$|\nabla\phi| \le c/R$.

$$\int_{B(p,2R)} \phi^2|\nabla^2 u|^2 \le \frac{1}{2}\int_{B(p,2R)} \phi^2\Delta|\nabla u|^2 - \phi^2\langle\nabla u, \nabla f\rangle$$
$$\le -\int_{B(p,2R)} \phi\nabla\phi\cdot\nabla|\nabla u|^2 - \phi^2\langle\nabla u, \nabla f\rangle$$
$$\le C\left[\int_{B(p,2R)} \phi^2 f^2 + \epsilon\phi^2|\nabla^2 u|^2 + (1+\frac{1}{\epsilon})|\nabla\phi|^2|\nabla u|^2\right]$$

Choose ϵ small enough to conclude the result. □

Theorem 2.3. *Let M^n be a complete Kahler manifold with $BK \ge 0$ and maximal volume growth. In addition, if the scalar curvature $S \le \frac{c}{r^{2+\epsilon}}$, then M is flat.*

Proof. By previous porposition, we can solve for u in which $\Delta u = s$. In particular, $|u| \le C$, $|\nabla u| = o(1/r)$. As $||Ric - \sqrt{-1}\partial\bar{\partial}u||$ is subharmonic, by Li's mean value inequality, for $R >> r(x)$,

$$||Ric - \sqrt{-1}\partial\bar{\partial}u||^2(x) \le \frac{C}{Vol(B(p,R))}\int_{B(p,R)} ||Ric - \sqrt{-1}\partial\bar{\partial}u||^2$$
$$\le \frac{C}{Vol(B(p,R))}\int_{B(p,R)} |\nabla^2 u|^2 + |Ric|^2 = O(\frac{1}{R^4}) \to 0.$$

Thus, $Ric = \sqrt{-1}\partial\bar{\partial}u$. To finish our proof, it suffices to show that u is constant.
For $n \ge 2$,

$$\int_{B(p,r)} (\partial\bar{\partial}u)^n = \int_{\partial B(p,r)} \bar{\partial}u \wedge (\partial\bar{\partial}u)^{n-1} = o(1/r)\cdot(\frac{1}{r^{2+\epsilon}})^{n-1}\cdot O(r^{2n-1}) \to 0.$$

Thus, $(\partial\bar{\partial}u)^n = 0$. Assuming M is stein, let $\Phi : M \to C^N$ be the embedding, $\Phi(p) = (z_1(p), z_2(p), ..., z_N(p))$. Let ϕ be the restriction of $\sum_{i=1}^N |z_i|^2$ on M. If u is non-constant, for each $c < 0$, $M_c = \{u < c\}$ is precompact. We may further assume M_c is smooth. Let x_0 be a point on ∂M_c such that ϕ attains maximum. So locally M_c is on one side of $\{\phi = \phi(x_0)\}$. Therefore $i\partial\bar{\partial}u$ is bounded below by the complex hessian of the distance function on the complex tangent space of ∂M_c at x_0. In particular, $i\partial\bar{\partial}u$ is strictly positive on it.

$$\partial\bar{\partial}e^u = e^u\partial\bar{\partial}u + e^u\partial u \wedge \bar{\partial}u$$

is then positive at x_0. However, on M

$$\int_{B(p,r)} (\partial\bar{\partial}e^u)^n = \int_{\partial B(p,r)} \bar{\partial}e^u \wedge (\partial\bar{\partial}e^u)^{n-1}$$
$$= \int_{\partial B(p,r)} e^{nu}\bar{\partial}u \wedge (\partial\bar{\partial}u + \partial u \wedge \bar{\partial}u)^{n-1} \to 0.$$

So, $(\partial\bar{\partial}e^u)^n \equiv 0$. Contradiction arised.
Remark: It has been proved later by Ni-Tam that M must be stein if it satisfies the assumption in this theorem. □

Corollary 2.4. *Suppose M is a Kahler manifold with maximal volume growth and $BK \geq 0$. Furthermore, if the scalar curvature $s \leq \frac{C}{r^2}$, then $\int_M Ric^n < \infty$.*

Proof. Solve for $i\partial\bar{\partial}u = Ric$.

$$\int_{B(p,R)} Ric^n = \int_{\partial B(p,R)} i\bar{\partial}u \wedge Ric^{n-1} \leq C' < \infty$$

Where C' is independent of R. $\square$

Theorem 2.5. *(Ni-Shi-Tam) The conclusion of Proposition 2.2 still hold if we only assume $Ric \geq 0$ and*

$$0 \leq f(x) \leq \frac{c}{1 + r(x)^2} \quad \text{and} \quad \frac{1}{V_p(r)} \int_{B(p,r)} f(x) \leq \frac{C}{1 + r^2}.$$

Corollary 2.6. *If M^n has $BK \geq 0$, and $Ric > 0$ at $p \in M$. Furthermore, if the scalar curvature satisfies $s \leq \frac{C}{r^2}$ and $\frac{1}{V_p(r)} \int_{B(p,r)} s \leq \frac{C}{r^2}$. Then M has maximal volume growth and $\exists c' = c'(M) > 0$ such that for r large*

$$\frac{1}{V_p(r)} \int_{B(p,r)} s \geq \frac{c'}{r^2}.$$

Proof. Solve for u where $i\partial\bar{\partial}u = Ric$ by Theorem above. $|\nabla u| \leq c/r$. Then

$$\begin{aligned} 0 < c < \int_{B(p,R)} Ric^n &= \int_{B(p,R)} i\partial\bar{\partial}u \wedge Ric^{n-1} \\ &= \int_{\partial B(p,R)} i\bar{\partial}u \wedge Ric^{n-1} \leq C \cdot \frac{1}{R} \cdot \frac{1}{R^{2n-2}} \cdot vol(\partial B(p,R)) \end{aligned}$$

which implies maximal volume growth.
Assume $r >> 1$,

$$\begin{aligned} 0 < c < \int_{B(p,r)} Ric^n &= \int_{\partial B(p,r)} i\bar{\partial}u \wedge Ric^{n-1} \\ &\leq \frac{C}{r^{2n-3}} \int_{\partial B(p,r)} Ric \wedge \omega^{n-1} \\ &= \frac{C}{r^{2n-3}} \int_{\partial B(p,r)} s. \end{aligned}$$

$\square$

Theorem 2.7. *(Chen-Zhu) Let M^n be a Kahler manifold $BK > 0$. Let $p \in M$. Then $\exists c = c(M,p) > 0$ such that*

$$vol(B(p,r)) \geq Cr^n.$$

Proof. Consider Busemann function b on M, where

$$b(x) = \lim_{r \to \infty} (r - d(x, \partial B(p,r))).$$

It is known that $|\nabla b| = 1$ *a.e.* and b is strictly plurisubharmonic in the support sense. By smoothing argument, for $R > 2$, $\exists c > 0$ such that

$$\begin{aligned} 0 < cR^{2n} \le \int_{B(p,R)} (i\partial\bar{\partial}b)^n (R-r)^{2n} &= 2n \int_{B(p,R)} (R-r)^{2n-1} \bar{\partial}b \wedge (i\partial\bar{\partial}b)^{n-1} \wedge \partial r \\ &\le C_n \int_{B(p,R)} (R-r)^{2n-1} (i\partial\bar{\partial}b)^{n-1} \wedge \omega \\ &\le C'_n \int_{B(p,R)} (R-r)^n \omega^n \le C'_n R^n V_p(R), \end{aligned}$$

which implies $Vol(B(p,r)) \ge c' r^n$, for all $r > 2$. □

3 Application of H$\ddot{o}$rmander L^2 estimate.

Theorem 3.1. *(Hörmander L^2 estimate): Let (M, ω) be a complete Kaehler manifold. $L \to M$ be holomorphic line bundle with Hermitian metric h (locally given by $e^{-2\phi}, |s|_h^2 = e^{-2\phi}$. Let Θ be the curvature (1,1)-form, $\Theta = 2i\partial\bar{\partial}\phi$. Let g be a smooth section in $\Lambda^{n,1} \otimes L$ (i.e. L valued (n,1) form.), with $\partial g = 0$. Assume $\Theta \ge \epsilon\omega$ where ϵ is positive function on M. If $\int_M |g|_h^2 \epsilon^{-1} < \infty$, then there exists $f \in \Gamma(\Lambda^{(n,0)} \otimes L)$ such that*

$$\partial f = g \ \ \text{and} \ \ \int_M |f|_h^2 \le \int_M \frac{|g|_h^2}{\epsilon}.$$

3.1 Some Applications of L^2-estimate

Theorem 3.2. *(Mok) Let M^n be a complete Kahler manifold with $BK > 0$ and maximal volume growth. Furthermore suppose the scalar curvature s satisfies $s \le \frac{c}{r^2}$, $r(x) = d(x, x_0)$ for some fixed point $x_0 \in M$, then there exists a non constant holomorphic function with polynomial growth on M.*

Proof. Due to the growth condition of scalar curvature s, one can solve u in which $i\partial\bar{\partial}u = Ric$. Furthermore, u satisfies

$$|u(x)| \le C \log(r(x) + 2).$$

Let $p \in M$, there exists a holomorphic chart $(z_1, ..., z_n)$ on $B(p, \delta)$ with $z_i(p) = 0$. Let φ be cut-off function with $\varphi = 1$ on $B(p, \delta/5)$ and $\varphi = 0$ outside $B(p, \delta/3)$. Now we look for a good weight function.

Lemma 3.3. *$i\partial\bar{\partial}(\lambda u + 2n\varphi \log(|z|^2)) \ge c\omega > 0$ on $B(p, \delta/3)$ for some $\lambda >> 1$, $c > 0$.*

Proof. Noted that $i\partial\bar{\partial} \log |z|^2 \ge 0$ in the current sense. Thus,

$$\begin{aligned} i\partial\bar{\partial}[\varphi \log |z|^2] &= i\varphi\partial\bar{\partial} \log |z|^2 + i\partial\bar{\partial}\varphi \cdot \log |z|^2 + i\partial\varphi \wedge \bar{\partial} \log |z|^2 + i\partial \log |z|^2 \wedge \bar{\partial}\varphi \\ &\ge -c, \end{aligned}$$

where c depends on the C^2 bounds of φ and δ. So for sufficiently λ, we have on $B(p, \delta/3)$

$$i\partial\bar{\partial}\left(\lambda u + 2n\varphi \log |z|^2\right) \geq c\omega.$$

$\square$

Take $L = T^{n,0}(M)$, choose metric $h = e^{-\psi} = e^{-(\lambda u + 2n\varphi \log |z|^2)}$. Thus, the curvature form $\Theta = -i\partial\bar{\partial}\log h \geq c\omega > 0$. Let ϕ be a smooth cut-off function such that $\phi = 1$ on $B(p, \delta/5)$ and $\phi = 0$ outside $B(p, \delta/4)$. Apply L^2 estimate to solve $\bar{\partial}g = \bar{\partial}(\phi z_1)$ with

$$\int_M |g|^2 e^{-\psi} \leq \frac{2}{c}\int_M |\bar{\partial}(\phi z_1)|^2 e^{-\psi} < \infty.$$

Noted that on $B(p, \delta/5)$, $e^{-\psi} = e^{-\lambda u}|z|^{-4n}$. Thus, it is not locally integrable which implies

$$g(p) = 0 \text{ and } dg(p) = 0.$$

Now $f = g - \phi z_1$ is holomorphic on M with $f(p) = 0$, $df(p) = d(\phi z_1) \neq 0$. f is thus non-constant and $f = g$ outside $B(p, \delta)$. It suffices to show that g is of polynomial growth. Let $y \in \partial B(p, R)$ in which $R > 100$,

$$C \geq \int_M |g|^2 e^{-\psi} \geq \int_{B(y,R/2)} |g|^2 e^{-\lambda u} \geq \int_{B(y,R/2)} |g|^2 (r+2)^{-c} \geq C' R^{-c} \int_{B(y,R/2)} |g|^2.$$

By Li's mean value inequality together with maximal volume growth condition, we conclude that $\exists C > 0$ independent of y such that

$$|g(y)|^2 \leq \frac{C_n}{V_y(R/2)} \int_{B(y,R/2)} |g|^2 \leq C R^{c-2n}.$$

$\square$

Theorem 3.4. *(Chen-Zhu) Let M^n be a Kahler manifold with $BK > 0$, then the scalar curvature s satisfies $\frac{1}{V_p(r)}\int_{B(p,r)} s \leq \frac{c}{r}$, where $c = c(M, p)$.*

Proof. Let $p \in M$. Let $(z_1, ...z_n)$ be a chart on $B(p, \delta)$. Consider the Busemann function b, it satisfies $i\partial\bar{\partial}b > \epsilon\omega$ for some $\epsilon > 0$ on $B(p, \delta)$. Define $\psi = \lambda b + 2n\phi \log |z|^2$, where ϕ is a cut-off function same as the proof above. λ is large enough such that $i\partial\bar{\partial}\psi \geq c\omega$ on $B(p, \delta)$.

Consider a smooth section on the canonical line bundle $u = \varphi dz_1 \wedge dz_2 \wedge ... \wedge dz_n$, where φ is a smooth function with compact support on $B(p, \delta/4)$ and equals to 1 on $B(p, \delta/5)$. By L^2 estimate, we obtain a nontrival holomorphic section θ on K_M such that $||\theta(p)|| = 1$ and

$$\int_M ||\theta||^2 e^{-\lambda b} < \infty.$$

Since the Busemann function is distance like function and $\Delta||\theta||^2 \geq 0$, by Li's mean value inequality [4], we have

$$|\theta(x)| \leq C_n e^{cr(x)}.$$

For $\delta > 0$, $\log(|\theta|^2+\delta)$ is smooth, and by directly computation,

$$\Delta \log(||\theta||^2+\delta) \geq \frac{s||\theta||^2}{||\theta||^2+\delta}.$$

Consider $M \times \mathbb{C}^2$ instead of M, positive green function exists and the volume growth is at least of order 4. For $\alpha, \beta >, \epsilon > 0$, $0 < \delta < 1$, for $(z,t) \in M \times \mathbb{C}^2$

$$\begin{aligned}
\int_{\beta > G(z) > \alpha} \frac{\tilde{s}||\theta||^2}{||\theta||^2+\delta}(G(z,\tilde{p})-\alpha)^{1+\epsilon} &\leq \int_{\beta > G(z) > \alpha} \Delta \log(||\theta||^2+\delta)(G-\alpha)^{1+\epsilon} \\
&= \int_{\beta > G(z) > \alpha} \log(||\theta||^2+\delta) \cdot \Delta (G-\alpha)^{1+\epsilon} \\
&\quad + \int_{G=\beta} \frac{\partial}{\partial n}[\log(||\theta||^2+\delta)](G-\alpha)^{1+\epsilon} \\
&\quad - \int_{G=\beta} (1+\epsilon)\log(||\theta||^2+\delta)(G-\alpha)^{\epsilon}\frac{\partial G}{\partial n}.
\end{aligned}$$

Noted that on $\{\beta > G > \alpha\}$,

$$\Delta (G-\alpha)^{1+\epsilon} = \epsilon(1+\epsilon)(G-\alpha)^{\epsilon-1}|\nabla G|^2 \geq 0.$$

So we get

$$\begin{aligned}
\int_{\beta > G(z) > \alpha} \log(||\theta||^2+\delta) \cdot \Delta (G-\alpha)^{1+\epsilon} &\leq \sup_{\beta > G > \alpha} \log(||\theta||^2+\delta) \cdot \int_{\beta > G > \alpha} \Delta (G-\alpha)^{1+\epsilon} \\
&\leq \sup_{\beta > G > \alpha} \log(||\theta||^2+\delta) \cdot \int_{G=\beta} (1+\epsilon)(G-\alpha)^{\epsilon}\frac{\partial G}{\partial n}.
\end{aligned}$$

Letting $\epsilon \to 0$ it follows

$$\begin{aligned}
\int_{\beta > G > \alpha} \frac{\tilde{s}||\theta||^2}{||\theta||^2+\delta}(G(z,p)-\alpha) \leq \sup_{\beta > G > \alpha} \log(||\theta||^2+\delta) \cdot \int_{G=\beta} \frac{\partial G}{\partial n} \\
+ \int_{G=\beta} \frac{\partial}{\partial n}[\log(||\theta||^2+\delta)](G-\alpha) \\
- \int_{G=\beta} \log(||\theta||^2+\delta)\frac{\partial G}{\partial n}.
\end{aligned}$$

Since G and $\frac{\partial G}{\partial n}$ are asymptotic to $\frac{c_n}{r^{2n-2}}$ and $\frac{c_n'}{r^{2n-1}}$ as $\beta \to \infty$, thus

$$|\int_{G=\beta}(G-\alpha)| \leq c_n \frac{r^{2n-1}}{r^{2n-2}} = c(n)r \to 0, \text{ as } \beta \to \infty.$$

and

$$\int_{G=\beta} \frac{\partial G}{\partial n} \to c(n) \text{ as } \beta \to \infty.$$

Since $||\theta(p)|| = 1$, letting $\beta \to \infty$ yield

$$\begin{aligned}
\int_{G > \alpha} \frac{\tilde{s}||\theta||^2}{||\theta||^2+\delta}(G(x,\tilde{p})-\alpha) &\leq c_n \left[\sup_{G > \alpha} \log(||\theta||^2+\delta) - \log(||\theta||^2(\tilde{p})+\delta)\right] \\
&\leq c_n \sup_{G > \alpha} \log(||\theta||^2+\delta).
\end{aligned}$$

Letting $\delta \to 0$ implies

$$\int_{G>2\alpha} \tilde{s}(z)G(z,\tilde{p}) \le c_n \sup_{G>\alpha} \log ||\theta||^2.$$

For $\alpha > 0$, let r_α be the maximum positive number such that $B(p, r_\alpha) \subset \{G > \alpha\}$. For M equiped with $Ric \ge 0$, the minimal postive Green's function satisfies

$$C_1^{-1} \int_{r(z)}^{\infty} \frac{t\ dt}{V_{\tilde{p}}(t)} \le G(x,p) \le C_1 \int_{r(z)}^{\infty} \frac{t\ dt}{V_{\tilde{p}}(t)}.$$

By volume comparsion,

$$G(z,\tilde{p}) \ge C_1^{-1} \int_{r(z)}^{\infty} \frac{t\ dt}{V_{\tilde{p}}(t)} \ge C_1^{-1} \int_{r(z)}^{\infty} \frac{tr^{2n}}{V_p(r)t^{2n}}\ dt \ge C_n \frac{r(z)^2}{V_{\tilde{p}}(r(z))}.$$

On the other hand,

$$G(z,\tilde{p}) \le C_1 \int_{r(z)}^{\infty} \frac{t\ dt}{V_{\tilde{p}}(t)} \le Cr(z)^4 \int_{r(z)}^{\infty} \frac{t\ dt}{V_{\tilde{p}}(r)t^4} = \frac{Cr(z)^2}{V_{\tilde{p}}(r(z))}.$$

As a result, there exists $C = C(n)$ such that for all $\alpha > 0$, r_α satisfies

$$C^{-1} \frac{r_\alpha^2}{V_{\tilde{p}}(r_\alpha)} \le \alpha \le \frac{Cr_\alpha^2}{V_{\tilde{p}}(r_\alpha)}.$$

Thus, if $G(z,\tilde{p}) > \alpha$,

$$\frac{C^{-1}r(z)^2}{V_{\tilde{p}}(r(z))} \ge G(z,\tilde{p}) > \alpha \ge C\frac{r_\alpha^2}{V_{\tilde{p}}(r_\alpha)}.$$

by the above inequality,

$$\frac{V_{\tilde{p}}(r(z))}{V_{\tilde{p}}(r_\alpha)} \le C_n \frac{r^2(z)}{r_\alpha^2}.$$

Since $B(p,r) \times B(0,r) \subset B(\tilde{p}, r) \subset B(p,r) \times B(0,r)$ for each $r > 0$,

$$\frac{V_p(r(z)/2)}{V_p(r_\alpha)} \le c_n \left(\frac{r_\alpha}{r(z)}\right)^4 \cdot \frac{V_p(r(z)/2)\cdot(r(z)/2)^4}{2^4 \cdot V_p(r_\alpha)\cdot r_\alpha^4} \le c_n \left(\frac{r_\alpha}{r(z)}\right)^4 \frac{V_{\tilde{p}}(r(z)/2)}{V_{\tilde{p}}(r_\alpha)} \le \frac{c_n r_\alpha^2}{r^2(z)}.$$

That is

$$r(z) \le c_n r_\alpha^2 \cdot \frac{V_p(r(z)/2)}{V_p(r_\alpha)}.$$

In both of the cases $r(z)/2 \le r_\alpha$ or $r(z)/2 \ge r_\alpha$, $r(z) \le c_n r_\alpha$. Thus $\{G > \alpha\} \subset B(\tilde{p}, c_n r_\alpha)$. Combine everythings, one can get

$$\begin{aligned} \frac{Cr_\alpha^2}{V_{\tilde{p}}(r_\alpha)} \int_{B(\tilde{p},r_\alpha)} \tilde{s}(z) &\le \int_{B(\tilde{p},r_\alpha)} \alpha \cdot \tilde{s}(z) \le \int_{G>\alpha} \alpha \cdot \tilde{s}(x) \le \int_{G>\alpha} G \cdot \tilde{s}(x) \\ &\le c_n \sup_{G>\alpha/2} \log ||\theta||^2 \le c_n \sup_{B(\tilde{p}, c_n r_{\alpha/2})} \log ||\theta||^2. \end{aligned}$$

On the other hand, r_α and $r_{\alpha/2}$ can be related in the following relation.

$$r_{\alpha/2}^2 \le C_n \alpha \cdot V_{\tilde{p}}(r_{\alpha/2}) \le C_n \frac{V_{\tilde{p}}(r_{\alpha/2})}{V_{\tilde{p}}(r_\alpha)} \cdot r_\alpha^2 \le C_n r_\alpha^2 \left(\frac{r_{\alpha/2}}{r_\alpha}\right)^{2n}.$$

So for any $\alpha > 0$,

$$\frac{r_\alpha^2}{V_{\tilde{p}}(r_\alpha)} \int_{B(\tilde{p},r_\alpha)} \tilde{s}\ d\tilde{V} \leq C(n,p) \sup_{B(\tilde{p},c_n' r_\alpha)} \log ||\theta||^2 \leq C(n,p)(r_\alpha + 1).$$

Projecting everythings back to M implies our desired result.

$\square$

4 Heat flow technique on Kaehler manifolds

Theorem 4.1. *(Ni-Tam) Let M^n be a complete noncompact Kahler manifold with nonnegative holomorphic bisectional curvature and let u be a continuous plurisubharmonic function on M satisfying*

$$|u|(x) \leq Ce^{ar(x)^2}$$

for some constants $a, C > 0$ where $r(x)$ is the distance of x from a fixed point. Let v be the solution of the heat equation with initial data u. There exists $T_0 > 0$ depending only on a and there exists $T_1 \in (0, T_0)$ such that the following are true.

1. *For $0 < t < T_0$, $v(\cdot, t)$ is a smooth plurisubharmonic function.*

2. *Let*

$$K(x,t) = \{w \in T_x^{1,0}(M) : v_{\alpha\bar{\beta}}(x,t)w^\alpha = 0,\ \forall\ \beta.\}$$

be the null space of $v_{\alpha\bar{\beta}}(x,t)$. Then for any $0 < t < T_1$, $K(x,t)$ is distribution on M. Moreover the distribution is invariant under parallel translations.

3. *If the holomorphic bisectional curvature is positive at some point, then $v(x,t)$ is strictly plurisubharmonic for all $0 < t < T_1$.*

Before we proceed to the proof, we first state some of its applications.

Corollary 4.2. *Let M^n be a complete Kahler manifold with nonnegative holomorphic bisectional curvature. Suppose M is of maximal volume growth, then M is stein.*

Proof. Assume first that M is simply connected. Let $b(x)$ be the Busemann function on M. By a result of Shen, if M^n has nonnegative Ricci curvature and maximal volume growth. Then the Busemann function is a exhaustion function. Consider the heat flow $v(t)$ with $v(0) = b(x)$. Let $u = v(t_0)$ where $t_0 < T_1$, then we obtain a smooth plurisubharmonic function on M. By heat kernel estimate, u is also a exhaustion function.

By the main theorem, $K(x,t)$ is parallel invariant, By de Rham decomposition, $M = N_1 \times M_1$ isometrically and holomorphically so that $u_{\alpha\bar{\beta}} \equiv 0$ when restricted on N_1 and $u_{\alpha\bar{\beta}} > 0$ on M_1.

Due to the non-collapsing condition, N_1 must be noncompact. On N_1, define $h = \log(1 +$

$|\nabla u|^2)$, h is plurisubharmonic. To see this, it suffices to show that $h_{\gamma\bar{\gamma}} \geq 0$ in normal coordinate. Let $F = |\nabla u|^2$,

$$\begin{aligned}
h_{\gamma\bar{\gamma}} &= \frac{1}{(1+F)^2}[(1+F)F_{\gamma\bar{\gamma}} - F_\gamma F_{\bar{\gamma}}] \\
&= \frac{1}{(1+F)^2}\left[(1+F)\left(\sum_\alpha u_{\gamma\alpha}u_{\bar{\gamma}\bar{\alpha}} + \sum_{\alpha,s} R_{\gamma\bar{\gamma}\alpha\bar{s}}u_s u_{\bar{\alpha}}\right) - \sum_\alpha (u_{\alpha\gamma}u_{\bar{\alpha}})\sum_\alpha (u_\alpha u_{\bar{\alpha}\bar{\gamma}})\right] \\
&\geq \frac{1}{(1+F)^2}\left(\sum_\alpha u_{\alpha\gamma}u_{\bar{\alpha}\bar{\gamma}} + \sum_{\alpha,s} R_{\gamma\bar{\gamma}\alpha\bar{s}}u_s u_{\bar{\alpha}}\right) \geq 0.
\end{aligned}$$

The Busemann function b is distance like, and $|\nabla b| = 1$ almost everywhere on M. Thus, $u = O(r(x))$ by Heat Kernel estimate. By gradient estimate, $|\nabla u| = O(1)$. And hence $h = o(\log(r(x))$. By three circle theorem [7], h is constant function on N_1 and hence $|\nabla u|$ is constant on N_1. By Bochner's formula, on N_1,

$$\begin{aligned}
0 = \frac{1}{2}\Delta|\nabla u|^2 &= |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta u\rangle + Ric(\nabla u, \nabla u) \\
&\geq |\nabla^2 u|^2.
\end{aligned}$$

∇u is parallel. Hence $J\nabla u$ is parallel. So $N_1 = N_1' \times \mathbb{C}$ as N_1 is simply connected. Repeating the process, we conclude that $N_1 = N \times \mathbb{C}^k$ for some k and N is compact. But Due to the non-collapsing assumption, N does not exist. So $M = \mathbb{C}^k \times M_1$ where u is a strictly plurisubharmonic exhaustion function on M_1. Thus, M_1 is stein and so is M.
If M is not simply connected, let $\tilde{M}$ be its universal cover, $\tilde{M}$ is of maximal volume growth and nonnegative Ricci curvature. By a result in [6], $\pi_1(M)$ is finite. Let f be a smooth strictly plurisubharmonic exhaustion function on $\tilde{M}$. Then

$$h(x) = \sum_{\pi^{-1}(x)} f(p)$$

is a strictly plurisubharmonic exhaustion function on M. Thus M is stein. $\square$

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